Linear Programming

Optimizing Linear Objective Function with Constraints

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1. Transportation Problem: Optimal Shipping Plan

A logistics company supplies goods from **three warehouses (W1, W2, W3)** to **four retail stores (S1, S2, S3, S4)**. The transportation cost per unit from each warehouse to each store is given in the table below. Each warehouse has a limited supply, and each store has a demand requirement. The goal is to minimize the total transportation cost.

To / From	S1	S2	S3	S4	Supply
W1	4	3	6	5	250

To / From	S1	S2	S3	S4	Supply
W2	2	5	3	4	300
W3	7	6	4	3	400
Demand	200	200	250	300	-

Decision Variables

Let x_{ij} be the number of units transported from warehouse iii to store j.

Objective Function

Minimize total transportation cost:

$Z = 4x_{11} + 3x_{12} + 6x_{13} + 5x_{14} + 2x_{21} + 5x_{22} + 3x_{23} + 4x_{24} + 7x_{31} + 6x_{32} + 4x_{33} + 3x_{34}$

Constraints

Supply Constraints

- $x_{11} + x_{12} + x_{13} + x_{14} \leq 250 \quad \text{(Warehouse W1)}$
- $x_{21} + x_{22} + x_{23} + x_{24} \leq 300 \quad \text{(Warehouse W2)}$
- $x_{31} + x_{32} + x_{33} + x_{34} \leq 400 \quad \text{(Warehouse W3)}$

Demand Constraints

- $x_{11} + x_{21} + x_{31} = 200 \quad \text{(Store S1)}$
- $x_{12} + x_{22} + x_{32} = 200 \quad \text{(Store S2)}$
- $x_{13} + x_{23} + x_{33} = 250 \quad \text{(Store S3)}$
- $x_{14} + x_{24} + x_{34} = 300 \quad \text{(Store S4)}$

Non-Negativity Constraints: $x_{ij} \geq 0 \quad \text{for all } i, j$

Answer

```
from scipy.optimize import linprog

# Cost coefficients
c_transport = [4, 3, 6, 5, 2, 5, 3, 4, 7, 6, 4, 3]

A_transport = [  # Coefficients for constraints (Supply + Demand)
    [1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0],  # W1 supply
    [0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0],  # W2 supply
    [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1],  # W3 supply
    [1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0],  # S1 demand
    [0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0],  # S2 demand
```

```python
    [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0],  # S3 demand
    [0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1],  # S4 demand
]

b_transport = [250, 300, 400, 200, 200, 250, 300]  # Supply & Demand constraints
bounds_transport = [(0, None)] * 12  # Non-negativity

res_transport = linprog(
    c_transport,
    A_ub = A_transport[:3],
    b_ub = b_transport[:3],
    A_eq = A_transport[3:],
    b_eq = b_transport[3:],
    bounds = bounds_transport,
    method='highs')
res_transport
```

```
        message: Optimization terminated successfully. (HiGHS Status 7: Optimal)
        success: True
         status: 0
            fun: 2850.0
              x: [ 5.000e+01  2.000e+02  0.000e+00  0.000e+00  1.500e+02
                   0.000e+00  1.500e+02  0.000e+00  0.000e+00  0.000e+00
                   1.000e+02  3.000e+02]
            nit: 6
          lower:  residual: [ 5.000e+01  2.000e+02  0.000e+00  0.000e+00
                              1.500e+02  0.000e+00  1.500e+02  0.000e+00
                              0.000e+00  0.000e+00  1.000e+02  3.000e+02]
                 marginals: [ 0.000e+00  0.000e+00  1.000e+00  1.000e+00
                              0.000e+00  4.000e+00  0.000e+00  2.000e+00
                              4.000e+00  4.000e+00  0.000e+00  0.000e+00]
          upper:  residual: [       inf        inf        inf        inf
                                    inf        inf        inf        inf
                                    inf        inf        inf        inf]
                 marginals: [ 0.000e+00  0.000e+00  0.000e+00  0.000e+00
                              0.000e+00  0.000e+00  0.000e+00  0.000e+00
                              0.000e+00  0.000e+00  0.000e+00  0.000e+00]
          eqlin:  residual: [ 0.000e+00  0.000e+00  0.000e+00  0.000e+00]
                 marginals: [ 4.000e+00  3.000e+00  5.000e+00  4.000e+00]
        ineqlin:  residual: [ 0.000e+00  0.000e+00  0.000e+00]
                 marginals: [-0.000e+00 -2.000e+00 -1.000e+00]
 mip_node_count: 0
```

```
mip_dual_bound: 0.0
       mip_gap: 0.0
```

Interpretation

- The **minimum total transportation cost is $2,850**.
- **Optimal shipment plan**:
  - **From W1 to S1:** 50 units
  - **From W1 to S2:** 200 units
  - **From W1 to S3 & S4:** 0 units
  - **From W2 to S1:** 150 units
  - **From W2 to S2:** 0 units
  - **From W2 to S3:** 150 units
  - **From W2 to S4:** 0 units
  - **From W3 to S1 & S2:** 0 units
  - **From W3 to S3:** 100 units
  - **From W3 to S4:** 300 units
- **Shadow prices (dual values)** for demand constraints:
  - **S1 = $4**, meaning if demand at S1 increases by 1 unit, total cost increases by $4.
  - **S2 = $3**, meaning an extra unit at S2 increases cost by $3.
  - **S3 = $5**, meaning an extra unit at S3 increases cost by $5.
  - **S4 = $4**, meaning an extra unit at S4 increases cost by $4.
- **Marginals**
  - **W1 = $0**, meaning increasing W1's supply doesn't impact cost.
  - **W2 = -$2**, meaning if W2's supply increased, costs could reduce by $2 per unit.
  - **W3 = -$1**, meaning if W3's supply increased, costs could reduce by $1 per unit.

2. Manufacturing Problem: Maximizing Profit (Product Mix)

A company produces **two types of products (A and B)** using **two machines (M1 and M2)**. The processing time (in hours per unit) and the profit per unit are given below. The company has a limited number of available hours for each machine. The objective is to maximize profit.

Product	M1 Hours per unit	M2 Hours per unit	Profit per unit ($)
A	3	2	50
B	5	4	80

- **Machine M1** has **600 hours** available.
- **Machine M2** has **500 hours** available.

Decision Variables

Let x_1 be the number of units of product A produced.
Let x_2 be the number of units of product B produced.

Objective Function

Maximize profit: $Z = 50x_1 + 80x_2$

Constraints

Machine Time Constraints

- $3x_1 + 5x_2 \leq 600 \quad \text{(M1 capacity)}$
- $2x_1 + 4x_2 \leq 500 \quad \text{(M2 capacity)}$

Non-Negativity Constraints: $x_1, x_2 \geq 0$

Answer

```
c_profit = [-50, -80]  # Coefficients (negative for maximization)

A_profit = [  # Machine constraints
    [3, 5],   # M1 constraint
    [2, 4],   # M2 constraint
]

b_profit = [600, 500]  # Available hours
bounds_profit = [(0, None), (0, None)]  # Non-negativity
```

```python
res_profit = linprog(
    c_profit,
    A_ub = A_profit,
    b_ub = b_profit,
    bounds = bounds_profit,
    method='highs')

res_profit
```

```
        message: Optimization terminated successfully. (HiGHS Status 7: Optimal)
        success: True
         status: 0
            fun: -10000.0
              x: [ 2.000e+02  0.000e+00]
            nit: 1
          lower:  residual: [ 2.000e+02  0.000e+00]
                 marginals: [ 0.000e+00  3.333e+00]
          upper:  residual: [       inf        inf]
                 marginals: [ 0.000e+00  0.000e+00]
          eqlin:  residual: []
                 marginals: []
        ineqlin:  residual: [ 0.000e+00  1.000e+02]
                 marginals: [-1.667e+01 -0.000e+00]
 mip_node_count: 0
 mip_dual_bound: 0.0
        mip_gap: 0.0
```

Interpretation

- **Maximum Profit**: \$10,000.0
- **Optimal production quantities**:

200.0 units of Product A and **0.0** units of Product B

- **Remaining capacities after allocating resources**:

Machine M1: Residual = **0.0** (Fully utilized) Machine M2: Residual = **100.0** (Unused)

- **Shadow price (dual value)**:

Machine M1 = **16.67**, meaning that if we had one more hour of M1, the profit would increase by **16.67**. Machine M2 = **0.0**, meaning extra hours for M2 won't increase profit

3. Manufacturing Problem: Minimizing Production Cost

A furniture company manufactures **chairs and tables**. The company has **limited resources of wood and labor and wants to minimize the total production cost**.

Product	Wood Required (cubic ft.)	Labor Required (hours)	Cost per unit ($)
Chair	5	2	30
Table	8	3	50

- Available wood: **800 cubic feet**

- Available labor: **300 hours**

Decision Variables

Let x_1 be the number of chairs produced.
Let x_2 be the number of tables produced.

Objective Function

Minimize cost: $Z = 30x_1 + 50x_2$

Constraints:

- $5x_1 + 8x_2 \leq 800 \quad \text{(Wood availability)}$
- $2x_1 + 3x_2 \leq 300 \quad \text{(Labor availability)}$

Non-Negativity Constraints: $x_1, x_2 \geq 0$

Answer

```
c_cost = [30, 50]  # Cost coefficients

A_cost = [  # Resource constraints
    [5, 8],  # Wood constraint
    [2, 3],  # Labor constraint
]

b_cost = [800, 300]  # Available resources
bounds_cost = [(0, None), (0, None)]  # Non-negativity

res_cost = linprog(
    c_cost,
```

```python
    A_ub = A_cost,
    b_ub = b_cost,
    bounds = bounds_cost,
    method = 'highs')

res_cost
```

```
        message: Optimization terminated successfully. (HiGHS Status 7: Optimal)
        success: True
         status: 0
            fun: 0.0
              x: [ 0.000e+00  0.000e+00]
            nit: 0
          lower:  residual: [ 0.000e+00  0.000e+00]
                 marginals: [ 3.000e+01  5.000e+01]
          upper:  residual: [       inf        inf]
                 marginals: [ 0.000e+00  0.000e+00]
          eqlin:  residual: []
                 marginals: []
        ineqlin:  residual: [ 8.000e+02  3.000e+02]
                 marginals: [-0.000e+00 -0.000e+00]
```

Interpretation

The **minimum production cost is $0.0**, meaning that the optimal decision is **not to produce any chairs or tables**.

Optimal production quantities:

- **Chairs: 0.0 units**
- **Tables: 0.0 units**
- The company should **not produce anything** to achieve the lowest cost.

Remaining resources:

- **Wood: 800.0 cubic feet unused**
- **Labor: 300.0 hours unused**
- Since no production takes place, all resources remain unused.

Shadow prices for wood and labor are both 0.0, meaning that increasing available resources would **not** change the optimal solution.

- This suggests that **there is no economic incentive to produce chairs or tables under the given cost structure**.