Simple Applications in Linear Problems

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Table of Contents

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Linear Programming (LP) is a mathematical technique used to **optimize** (maximize or minimize) a linear objective function subject to linear constraints, aiming to find the best outcome under given conditions. Key components include the **Objective Function** (the goal to optimize), **Constraints** (limitations or restrictions), and **Decision Variables** (the choices to adjust). LP problems generally involve minimizing or maximizing $Z = c_1 \times 1 + \ldots + c_n \times n$ subject to linear inequalities and non-negativity constraints.

Applications include **Production Scheduling**, where the objective is to optimize the production mix to maximize profit or minimize cost based on resource constraints, and **Transportation Problems**, which aim to minimize total shipping cost from sources to destinations subject to supply and demand constraints.

Solving methods range from the graphical method for problems with 2 variables to the simplex algorithm, special transportation methods (like `Northwest Corner`, `Least Cost`, `MODI`), and the use of optimization software or solvers.

Graphical Method (2 variables)

A small workshop makes two types of furniture: chairs and tables. Each chair requires 2 hours of carpentry and 1 hour of painting. Each table requires 1 hour of carpentry and 1 hour of painting. The workshop has 6 hours of carpentry time and 4 hours of painting time available each day. Each chair gives a profit of \$30, and each table gives a profit of \$20.

Task:

- Formulate the problem as a linear program.
- Plot the feasible region and determine the optimal number of chairs and tables to maximize profit using a graphical method.

Answer

Formulation of the Problem

Let:

- x = number of chairs produced
- y = number of tables produced

Objective: Maximize profit

Maximize $Z = 30x + 20y$

Subject to constraints:

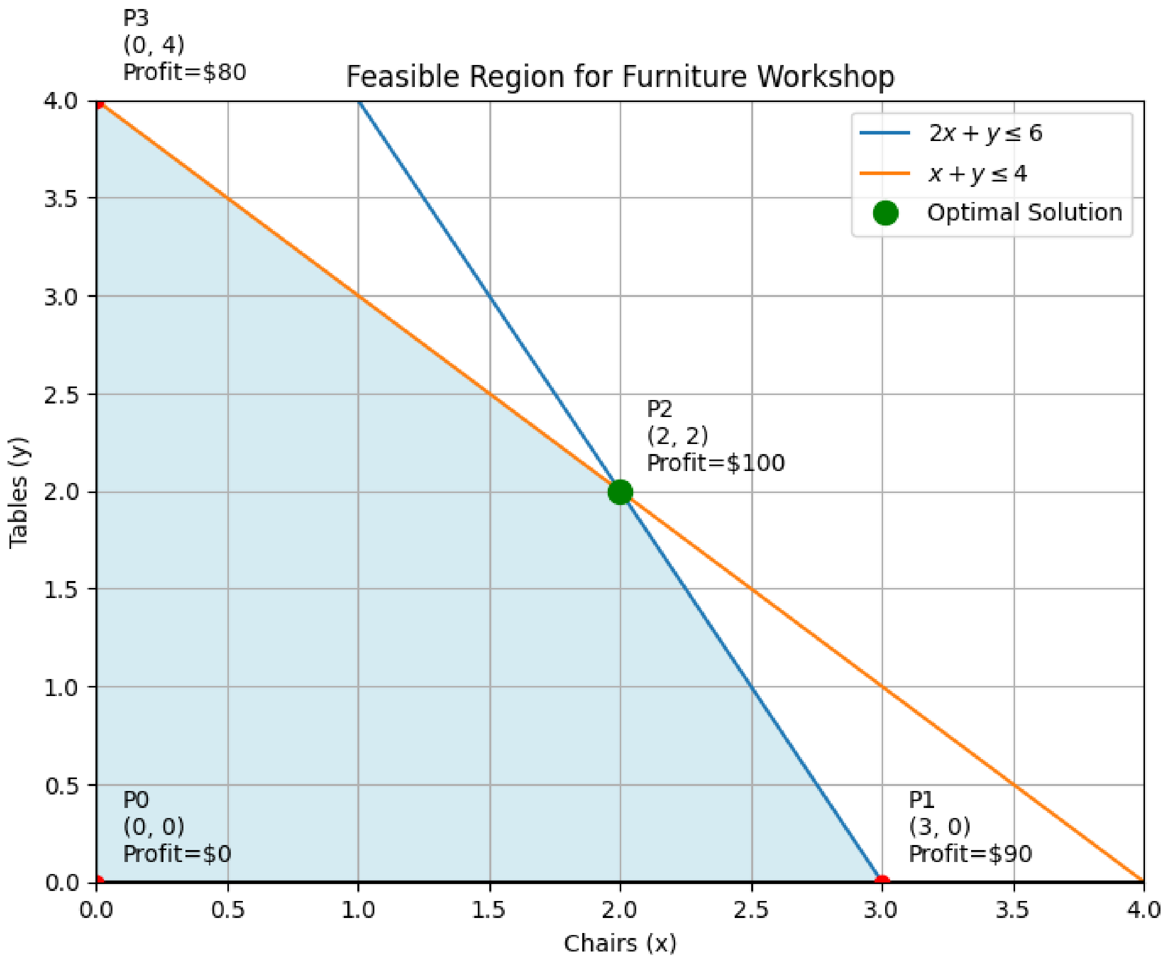
- Carpentry time:
  $2x + y \leq 6$
- Painting time:
  $x + y \leq 4$
- Non-negativity:
  $x \geq 0, \quad y \geq 0$

Plot the Feasible Region

We plot the constraints: - $2x + y = 6$ => (x-intercept at 3, y-intercept at 6) - $x + y = 4$ => (x-intercept at 4, y-intercept at 4)

```
import numpy as np
import matplotlib.pyplot as plt

# Define the constraints
x = np.linspace(0, 4, 100)

# Constraint lines
y1 = (6 - 2*x)  # from 2x + y <= 6
y2 = (4 - x)    # from x + y <= 4

# Plotting
plt.figure(figsize=(8,6))
plt.plot(x, y1, label=r'$2x + y \leq 6$')
plt.plot(x, y2, label=r'$x + y \leq 4$')

# Feasible region
plt.fill_between(
    x,
    0,
    np.minimum(y1, y2),
    where=(np.minimum(y1, y2)>=0),
    color='lightblue',
    alpha=0.5)

# Label axes
plt.xlim(0, 4)
plt.ylim(0, 4)
plt.xlabel('Chairs (x)')
plt.ylabel('Tables (y)')
plt.title('Feasible Region for Furniture Workshop')
plt.axhline(0, color='black')
plt.axvline(0, color='black')
plt.legend()

# Define corner points manually
points = np.array([
    [0, 0],
    [3, 0],    # from 2x + y = 6
```

```
    [2, 2],    # intersection point
    [0, 4],    # from x + y = 4
])

# Objective function: 30x + 20y
profits = [30*x + 20*y for x, y in points]

# Find max profit
max_profit_index = np.argmax(profits)
optimal_point = points[max_profit_index]
optimal_profit = profits[max_profit_index]

# Plot points
for i, (pt, profit) in enumerate(zip(points, profits)):
    plt.plot(pt[0], pt[1], 'ro')
    plt.text(
        pt[0]+0.1,
        pt[1]+0.1,
        f'P{i}\n({pt[0]}, {pt[1]})\nProfit=${profit}')

# Highlight optimal point
plt.plot(
    optimal_point[0],
    optimal_point[1], 'go',
    markersize=10,
    label='Optimal Solution')
plt.legend()

plt.grid(True)
plt.show()

# Print solution
print(f"Optimal production plan: {int(optimal_point[0])} chairs and {int(optimal_point[1])}
print(f"Maximum profit: ${optimal_profit}")
```




```
Optimal production plan: 2 chairs and 2 tables
Maximum profit: $100
```

Simplex Algorithm (via `scipy.optimize.linprog`)

A factory produces 3 products: A, B, and C. Each requires machine hours on 2 machines: M1 and M2.

Product	Profit	M1 Hours	M2 Hours
A	$40	2	1
B	$30	1	2
C	$20	1	1

- M1 is available for 100 hours/week.
- M2 is available for 80 hours/week.

Task:

- Formulate and solve using the Simplex algorithm via `scipy.optimize.linprog`.
- Determine how many units of A, B, and C to produce to maximize profit.

Answer

Formulate the Linear Program

Let: - x_1 = number of units of product A - x_2 = number of units of product B - x_3 = number of units of product C

Objective: Maximize profit

Maximize $Z = 40x_1 + 30x_2 + 20x_3$

But `scipy.optimize.linprog` minimizes by default, so we minimize (-Z):

Minimize $Z = -40x_1 - 30x_2 - 20x_3$

Subject to constraints:

- Machine 1 hours:
  $2x_1 + 1x_2 + 1x_3 \leq 100$
- Machine 2 hours:
  $1x_1 + 2x_2 + 1x_3 \leq 80$
- Non-negativity:
  $x_1, x_2, x_3 \geq 0$

```
from scipy.optimize import linprog

# Coefficients for the objective function (negative because linprog minimizes)
c = [-40, -30, -20]

# Coefficients for the inequalities
A = [
    [2, 1, 1],  # M1 constraint
    [1, 2, 1],  # M2 constraint
]

# Right-hand side values
b = [100, 80]
```

```python
# Bounds for variables (all must be >= 0)
x_bounds = (0, None)
bounds = [x_bounds, x_bounds, x_bounds]

# Solve the linear program
result = linprog(c, A_ub=A, b_ub=b, bounds=bounds, method='highs')

# Print results
if result.success:
    x1, x2, x3 = result.x
    max_profit = -result.fun
    print(f"Optimal production plan:")
    print(f"Product A: {x1:.2f} units")
    print(f"Product B: {x2:.2f} units")
    print(f"Product C: {x3:.2f} units")
    print(f"Maximum profit: ${max_profit:.2f}")
else:
    print("No solution found.")
```

```
Optimal production plan:
Product A: 40.00 units
Product B: 20.00 units
Product C: 0.00 units
Maximum profit: $2200.00
```

Transportation Method - Northwest Corner Rule (manual or pandas/numpy)

A company has 3 factories (S1, S2, S3) and 4 distribution centers (D1, D2, D3, D4).

Supplies:

- S1: 30 units
- S2: 40 units
- S3: 20 units

Demands:

- D1: 20 units
- D2: 30 units
- D3: 25 units
- D4: 15 units

Cost Matrix:

	D1	D2	D3	D4
S1	8	6	10	9
S2	9	7	4	2
S3	3	4	2	5

Task: - Use the Northwest Corner Method to construct an initial feasible solution manually or via a custom function in Python. - Display the allocation matrix and compute the total transportation cost.

Answer

Apply Northwest Corner Method (Manual)

We start from the top-left corner and allocate as much as possible.

- Start at (`S1`, `D1`):
  - Demand `D1 = 20`
  - Supply `S1 = 30`
  - Allocate **20** units (fulfills `D1`).
  - Remaining supply `S1 = 10`, `D1` fulfilled.
- Move to (`S1`, `D2`):
  - Demand `D2 = 30`
  - Supply `S1 = 10`
  - Allocate **10** units (uses all of `S1`).
  - Remaining demand `D2 = 20`, `S1` exhausted.
- Move to (`S2`, `D2`):
  - Demand `D2 = 20`
  - Supply `S2 = 40`
  - Allocate **20** units.
  - Remaining supply`S2 = 20`, `D2` fulfilled.
- Move to (`S2`, `D3`):
  - Demand `D3 = 25`
  - Supply `S2 = 20`
  - Allocate **20** units (uses all of `S2`).
  - Remaining demand `D3 = 5`, `S2` exhausted.
- Move to (`S3`, `D3`):

- Demand `D3 = 5`
- Supply `S3 = 20`
- Allocate **5** units.
- Remaining supply `S3 = 15`, `D3` fulfilled.

- Move to (`S3`, `D4`):
  - Demand `D4 = 15`
  - Supply `S3 = 15`
  - Allocate **15** units (both `S3` and `D4` are fulfilled).

Allocation Matrix

	D1	D2	D3	D4
S1	20	10	0	0
S2	0	20	20	0
S3	0	0	5	15

Compute Total Transportation Cost

$(20 \times 8) + (10 \times 6) + (20 \times 7) + (20 \times 4) + (5 \times 2) + (15 \times 5) = 525$

Python

```
import numpy as np

# Cost matrix
cost = np.array([
    [8, 6, 10, 9],
    [9, 7, 4, 2],
    [3, 4, 2, 5]
])

supply = [30, 40, 20]
demand = [20, 30, 25, 15]

# Initialize allocation matrix
alloc = np.zeros((3,4), dtype=int)

i, j = 0, 0

while i < 3 and j < 4:
    alloc_qty = min(supply[i], demand[j])
```

```
    alloc[i, j] = alloc_qty
    supply[i] -= alloc_qty
    demand[j] -= alloc_qty

    if supply[i] == 0:
        i += 1
    else:
        j += 1

# Calculate total cost
total_cost = np.sum(alloc * cost)

print("Allocation Matrix:")
print(alloc)
print(f"Total Transportation Cost: ${total_cost}")
```

```
Allocation Matrix:
[[20 10  0  0]
 [ 0 20 20  0]
 [ 0  0  5 15]]
Total Transportation Cost: $525
```
